

Two-time scale subordination in physical processes with long-term memory

Aleksander Stanislavsky ^{a,*}, Karina Weron ^b

^a*Institute of Radio Astronomy, Ukrainian National Academy of Sciences,
4 Chervonopraporna St., 61002 Kharkov, Ukraine*

^b*Institute of Physics, Wrocław University of Technology,
Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland*

Abstract

We use the two-time scale subordination in order to describe dynamical processes in continuous media with a long-term memory. Our consideration touches two physical examples in detail. First we study a temporal evolution of the species concentration for the trapping reaction in which a diffusing reactant is surrounded by a sea of randomly moving traps. The analysis is based on the random-variable formalism of anomalous diffusive processes. We find that the empirical trapping-reaction law, according to which the reactant concentration decreases in time as a product of an exponential and a stretched exponential function, can be explained by the two-time scale subordination of random processes. Another example is connected with a state equation for continuous media with memory. If the pressure and the density of a medium are subordinated in two different random processes, then the ordinary state equation becomes fractional with two time scales. This allows one to arrive at the state equation of Bagley-Torvik type.

Key words: Subordination, Nonexponential relaxation, Trapping reaction, Anomalous diffusion, Continuous medium with memory

PACS: 02.50.-r, 82.40.-g, 02.50.Ey

1 Introduction

The problem of trapping reactions has a long and active history tracked in the literature (see, for example, [1,2,3]). The reaction dynamics has been also studied over the past decade [4,5,6,7,8]. In the traditional version a reactant

* *E-mail address:* alexstan@ri.kharkov.ua

(A) walks in a medium randomly doped with static traps (B) so that when they meet, the reactant disappears ($A + B \rightarrow B$). This has served as a starting point for the formulation of several models describing the behavior of more complex systems. An important variation of the basic trapping problem is that the traps become diffusive too. When both species (A and B) are (sub)diffusive, the temporal evolution of the system looks like a random motion (of walker A) in a random sea of traps (trapping random walkers B). One approach for the analysis of such processes is based on the continuous time random walk (CTRW) in which space random jumps follow among subsequent random waiting times [9,10]. Another approach is stated on the fractional diffusion equation, which describes the probability density of finding the particle at position x at time t [11,12]. Recently, the relationship between the CTRW, stable distributions and the fractional calculus has been established exactly [13,14]. It turns out that if the CTRW is represented as a subordination of a space random (Brownian or Lévy) process by a inverse-time α -stable Lévy process, the probability density of a walker position is governed by a fractional Fokker-Planck equation. The approach was also extended on anomalous relaxation processes [15,16,17,18]. The aim of the given paper is to present a development of the subordination formalism to the complex relaxation processes. We are going to consider a time evolution of the concentration in the trapping reactions when the relaxation function takes the form of a product of an exponential and a stretched exponential function. The experiment supporting this empirical law is described in the paper of Djordjević [20]. The author studied the problem of an electron transfer from the methyl viologen radical cations to the colloidal platinum particle in a water solution. In that reaction the monocations MV^+ became the dications MV^{2+} and the dependence of MV^+ concentration on time appeared to follow the combined exponential-stretched-exponential decay law. In particular the experiment has stated the relaxation curve following $\log C(t) = -0.0011825t - 0.0608t^{0.6}$ such that the cautious estimation of the exponent of the second term is 0.56 ± 0.06 . Next, Djordjević also noticed that the contribution of the second term is about 90% of $\log C(t)$. One of the aims of his paper was to verify experimentally the fact of factorization. The task was realized with success. We believe that this experiment is good, and the data fitting gives the physical dependence clearly.

The paper is organized as follows. In Sec. 2 we present important features of subordinate random processes with different subordinators. They are directly connected with the anomalous diffusion. In the dependence of the subordinator form the corresponding subordinate processes have a different evolution in time. In our analysis we use the peculiarity of relaxation functions. From the paper [18] it is known what subordinator leads to the stretched exponential law. In Sec. 3 we show that for the relaxation law to take the combined form, the subordinator should contain two internal-time scales. One of them gives the exponential relaxation, and another contributes the stretched exponential function. In the spirit of the consideration, Section 4 is devoted to the study of

continuous media with a long-term memory. We consider such media that are characterized by the two time scales in the state equation. Section 5 presents a summary of results.

2 Stretched exponential response and its subordinator

If a complex physical system consisting of identical objects (dipoles, charges and so on) undergoes an irreversible transition, say, from state A to state B at random instances of time, then the transition can be characterized by the probability in the form

$$\Pr(\theta \geq t) = \exp \left(- \int_0^t r(y) dy \right),$$

where the non-negative quantity $r(y)$ denotes the system's transition rate, i.e., the transition probability intensity for transition of the system as a whole. This formula simply follows from a two-stage master equation (see, for example, [21]). The probability $\Pr(\theta \geq t)$ shows that a considered object will remain in state A until time t . In general, the transition rate $r(y)$ is time-dependent because of random impacts affecting each object. If one knows the explicit form of $r(y)$, the value $\Pr(\theta \geq t)$ can be derived. In the simplest case, when the quantity $r(y) = b_0 = \text{const}$ is time-independent, the above formula recovers the classical exponential relaxation law

$$\phi(t) = \Pr(\theta \geq t) = e^{-\omega_p t},$$

where $\omega_p = b_0$ is a characteristic material constant. The exponential law, however hardly ever found in nature, is widely accepted for description of various relaxation data. Such a model of relaxation assumes that the relaxation rate (inverse of the relaxation time) of each object is constant. Although it may be different from one object to others, the mean of the effective relaxation rate (representing the inherent stochastic nature of the relaxation process) has to take a finite value. Nevertheless, the extensive experimental investigations in a wide frequency-time domain have shown [22,23] relatively large deviations from the exponential relaxation law. It has been found that for many materials the relaxation response follows the stretched exponential pattern

$$\phi(t) = e^{-(\omega_p t)^\alpha}, \quad 0 < \alpha < 1.$$

In this case, on contrary, the system's transition rate is essentially time dependent, and the mean of the effective relaxation rate becomes infinite [24].

The time dependence of the system's transition rate as well as the statistical properties of the effective relaxation rate clearly depend on the defect-diffusion relaxation mechanism in the system under consideration [10]. Hence, the exponential and the nonexponential relaxation can be modeled by means of the diffusive behavior of the systems as a whole. Of course, the exponential relaxation and nonexponential one correspond to different diffusion processes.

Consider a sequence T_i , $i = 1, 2, \dots$ of non-negative, independent, identically distributed random variables which represent waiting-time intervals between subsequent jumps of a particle. The random time interval of n jumps in space is written as

$$T(n) = \sum_{i=1}^n T_i, \quad T(0) = 0. \quad (1)$$

The number of the particle jumps performed up to time $t > 0$ is determined by the largest index n for which the sum of n interjump time intervals does not exceed the observation time t

$$N_t = \max\{n : T(n) \leq t\}.$$

The process N_t is called the renewal process, or else, the counting process. The position of the particle (i.e., the total distance) after N_t jumps becomes

$$R(N_t) = \sum_{i=1}^{N_t} R_i, \quad (2)$$

where R_i are independent, identically distributed variables giving both the length and the direction of the i -th jump. The process (2) is just known as the CTRW.

Assume that the time intervals T_i belong to the domain of attraction of a completely asymmetric stable distribution with the index $0 < \alpha < 1$. The generalization of the central limit theorem yields the continuous limit of the random sum (1), namely

$$a^{-1/\alpha} T([a\tau]) \xrightarrow{d} U(\tau) \quad \text{as} \quad a \rightarrow \infty,$$

where $U(\tau)$ is a strictly increasing α -stable Lévy process, $a > 0$ parameter, $[x]$ denotes the integer part of x and " \xrightarrow{d} " means "tends in distribution". Similarly, let the jumps R_i belong to the domain of attraction of a γ -stable distribution $S_{\gamma,\beta}(x)$, $0 < \gamma \leq 2$, $|\beta| \leq 1$ so that

$$a^{-1/\gamma} R([a\tau]) \xrightarrow{d} X(\tau) \quad \text{as} \quad a \rightarrow \infty,$$

where $X(\tau)$ is a γ -stable Lévy process. If $\gamma = 2$, the latter process is the classical Brownian motion. Both the process $U(\tau)$ and the process $X(\tau)$ are indexed by the internal time τ . The time is not the real, physical time. In order to find a particle position at the observable time t , we have to introduce the notion of the inverse-time α -stable Lévy subordinator V_t relating the internal τ and the observable t times

$$a^{-\alpha} N_{at} \xrightarrow{d} V_t = \inf\{\tau : U(\tau) > t\} \quad \text{as} \quad a \rightarrow \infty. \quad (3)$$

Then the continuous limit of the CTRW process obtains the following form

$$a^{-\alpha/\gamma} R(N_{at}) \approx (a^\alpha)^{-1/\gamma} R([a^\alpha V_t]) \xrightarrow{d} X(V_t) \quad \text{as} \quad a \rightarrow \infty \quad (4)$$

known as the anomalous diffusion process. In other words, the scaling limit of the CTRW leads to the anomalous diffusion process $X(V_t)$ in which the parent process $X(\tau)$, replacing the random discrete-time jumps $R(n)$, is subordinated by the directing process V_t , replacing the counting process N_t . It has been rigorously proved that the probability density of $X(V_t)$ is the solution of well-known fractional diffusion equation [13,14]. As the processes $X(\tau)$ and V_t are independent, following the total probability formula, the probability density $p_\alpha(x, t)$ of $X(V_t)$ can be written as

$$p_\alpha(x, t) = \int_0^\infty f_\alpha(\tau, t) g(x, \tau) d\tau,$$

where $f_\alpha(\tau, t)$ and $g(x, \tau)$ are the probability density of V_t and $X(\tau)$ respectively. Similarly, the Fourier transform $\tilde{p}_\alpha(k, t) = \langle \exp(ikX(V_t)) \rangle$ and the Laplace transform $\bar{p}_\alpha(k, t) = \langle \exp(-kX(V_t)) \rangle$ are given by

$$\tilde{p}_\alpha(k, t) = \int_0^\infty f_\alpha(\tau, t) \tilde{g}(k, \tau) d\tau, \quad (5)$$

$$\bar{p}_\alpha(k, t) = \int_0^\infty f_\alpha(\tau, t) \bar{g}(k, \tau) d\tau. \quad (6)$$

Here $k > 0$ has the physical meaning of a wave number.

The analysis of the properties of the diffusion front (i.e., the asymptotic distribution of the particle position in time t) allows one to determine [16,17] the one-parameter Mittag-Leffler relaxation function (corresponding to the Cole-Cole law in the frequency domain), namely

$$\phi(t) = \langle e^{-kX(V_t)} \rangle = E_\alpha(-c_{\alpha,\gamma} k^\gamma t^\alpha), \quad (7)$$

where $E_\alpha(x) = \sum_{k=0}^{\infty} x^k / \Gamma(n\alpha + 1)$ is the Mittag-Leffler function, $c_{\alpha,\gamma}$ is constant ($\omega_p = (c_{\alpha,\gamma} k^\gamma)^{1/\alpha}$). For the relaxation function to take another form, the subordinator of the process $X(t)$ should be changed.

With that end in view the papers [15,18] show that if the subordinator is a fully asymmetric Lévy α -stable process with the self-similar property

$$V_{bt}^{(\alpha)} \stackrel{d}{=} (bt)^{1/\alpha} V_1^{(\alpha)}, \quad (8)$$

where “ $\stackrel{d}{=}$ ” reads “equal in law”, the type of relaxation remains simple exponential

$$\left\langle e^{-kV_t^{(\alpha)}} \right\rangle = e^{-c_\alpha k^\alpha t}, \quad 0 < \alpha < 1. \quad (9)$$

This means that there are different scenarios leading to the exponential term in the relaxation function. It can be the Brownian as well as a deterministic process.

Nevertheless, if a new subordinator $\overline{V}_t^{(\alpha)}$ is defined as [18]

$$\overline{V}_t^{(\alpha)} \stackrel{d}{=} ct V_1^{(\alpha)}, \quad c > 0, \quad (10)$$

then the relaxation law decays otherwise. This process is constructed on the α stable Lévy process $V_t^{(\alpha)}$ multiplying both sides of its self-similarity property

$$V_{ct}^{(\alpha)} \stackrel{d}{=} (ct)^{1/\alpha} V_1^{(\alpha)}, \quad c > 0, \quad (11)$$

by $(ct)^{1-1/\alpha}$. Both processes $V_t^{(\alpha)}$ and $\overline{V}_t^{(\alpha)}$ have different properties. The old one $V_t^{(\alpha)}$ is an $1/\alpha$ self-similar Lévy process, and this means that it is an α stable Lévy process. While the new one is an α stable process, not Lévy (i. e. it does not have independent increments). Note that any Lévy process is a process with independent and stationary increments, whereas the α stable is a process for which all finite dimensional distributions are stable. Hence, the α stable Lévy process $V_t^{(\alpha)}$ has independent and stationary increments as well as all finite dimensional distributions stable. It is a broad class of random processes (see Table in [18]).

It would be useful to notice that the self-similarity of the process $V_t^{(\alpha)}$ can be also written as

$$V_{at}^{(\alpha)} \stackrel{d}{=} a^{1/\alpha} V_t^{(\alpha)}, \quad a > 0,$$

where the constant a is dimensionless. Then the corresponding property for the new process $\bar{V}_t^{(\alpha)}$ takes the form

$$\bar{V}_{at}^{(\alpha)} \stackrel{d}{=} a\bar{V}_t^{(\alpha)}, \quad a > 0.$$

This once again confirms their different behavior.

Now consider the subordinated process $X(\bar{V}_t^{(\alpha)})$ in which for simplicity a random process $X(t)$ is ordinary Brownian, and the directing process is $\bar{V}_t^{(\alpha)}$. Let the subordinated process $X(\bar{V}_t^{(\alpha)})$ with such a subordinator have a probability density $\hat{p}_\alpha(x, t)$. The relationship between the probability densities of parent and directing processes is expressed in the integral form

$$\hat{p}_\alpha(x, t) = \int_0^\infty p_1(x, t\xi) dg_\alpha(\xi), \quad (12)$$

where the probability distribution $g_\alpha(\xi)$ is described by the Laplace transform

$$G_\alpha(s) = \int_0^\infty \exp\{-s\xi\} dg_\alpha(\xi) = \exp\{-(As)^\alpha\}$$

with $s \geq 0$ and $A > 0$. The probability density $p_1(x, t)$ describes the normal diffusion. The anomalous diffusion $X(\bar{V}_t^{(\alpha)})$ gives the stretched exponential function of relaxation

$$\left\langle e^{-k\bar{V}_t^{(\alpha)}} \right\rangle = e^{-c_\alpha t^\alpha k^\alpha}, \quad 0 < \alpha < 1, \quad c_\alpha > 0. \quad (13)$$

In the next section we will demonstrate how the combined laws of relaxation can appear in the evolution of relaxing physical systems due to a further development of the subordination approach.

3 Subordination by two random processes

Let two independent sequences $\{T_i^{(1)}\}$ and $\{T_i^{(2)}\}$, $i = 1, 2, \dots$ consist of non-negative, independent and identically distributed random variables. The random variables $T_i^{(1)}$ and $T_i^{(2)}$ are attracted to α -stable laws with different indices. Let the temporal variables $T^{(1)}(n)$ and $T^{(2)}(n)$ be a sum of the sequences of time intervals, $T_i^{(1)}$ and $T_i^{(2)}$ respectively. The counting process N_t describes the number of the particle jumps performed up to time t . The position of the

particle after n jumps is a sum of the jumps R_i . The total distance reached by a particle during time t is defined by the number of the jumps by means of the counting process N_t . In this case the process N_t is something like the operational time. Since the sequences of the time intervals are independent on each other, their subordinators will be such too. To pass to the continuous limit, the probability density of the obtained subordinate process should be expressed as

$$\begin{aligned} p_{\alpha,\beta}(x,t) &= \\ &= \int_0^\infty \int_0^\infty f_\alpha(\tau_2,t) g\left(x, \frac{\tau_1 + \tau_2}{2}\right) f_\beta(\tau_1,t) d\tau_1 d\tau_2, \end{aligned} \quad (14)$$

where the variables τ_1 and τ_2 correspond to the two-time scale subordination for the parent process with the probability density $g(x, \tau)$. Here, we also have an anomalous diffusion.

In the case of the nonbiased random walk the process $X(\tau)$ belonging to the class of γ -stable Lévy process has the following characteristic function

$$\langle e^{ikX(\tau)} \rangle = e^{-c_\gamma k^\gamma \tau}, \quad c_\gamma > 0.$$

If the parent process is directed by the two subordinators, then the relaxation function takes the form

$$\begin{aligned} \phi_{\text{two}}(t) &= \\ &= \int_0^\infty \int_0^\infty e^{-c_\gamma k^\gamma (\tau_1 + \tau_2)/2} f_\alpha(\tau_1,t) f_\beta(\tau_2,t) d\tau_1 d\tau_2. \end{aligned} \quad (15)$$

As a result, we obtain a combined process.

Now, we look at the trapping reaction with the randomly moving traps. According to the experimental data [20], one can prepare the corresponding experimental system so that after activating the traps, the reactant concentration will relax in correspondence with a combined law. This law may be a product of some (in the simplest case, two) well-known relaxation laws (such as exponential, stretched exponential and so on). From the above consideration it follows that the relaxation function for the trapping reaction can be expressed in terms of

$$\begin{aligned} \phi_{\text{TR}}(t) &= \langle e^{-kV_t^{(\alpha)} - k\bar{V}_t^{(\beta)}} \rangle = \langle e^{-kV_t^{(\alpha)}} \rangle \langle e^{-k\bar{V}_t^{(\beta)}} \rangle = \\ &= e^{-c_\alpha k^\alpha t} \cdot e^{-c_\beta t^\beta k^\beta}, \quad 0 < \alpha, \beta < 1, \quad c_\alpha, c_\beta > 0. \end{aligned} \quad (16)$$

To sum up, the relaxation function is written as a simple product of the ordinary exponential and stretched exponential functions. Two random processes are present at the reaction: 1) reactant walks in a medium; 2) the traps appear randomly. If these processes are independent, then their contribution in the relaxation function can be a product of two relaxation dependencies. One of them gives a simple exponent, and another tends to a stretched exponential function. The indices α and β permit one to distinguish the random processes, but we do not define concretely what component (reactant or traps) leads to, for example, the stretched-exponent contribution because this depends on the experimental situation. We describe the most general picture of this phenomena from the probabilistic formalism of limit theorems.

Why is subordination important here? Because the reactant walks is (sub)diffusive, and (sub)diffusion is a result of subordination of one random process by another [25]. In like manner this relates to traps. The nontriviality of our analysis is due to the fact that the effective stochastic process in time at a fixed point in space becomes non-Markovian. For a non-Markovian process, calculation of any history dependent quantity such as functionals of trajectories is extremely hard barring a few special cases (see more details, for example, in [25]). Over past decade the probabilistic formalism of limit theorems has made a very great advance on the analysis of non-Markovian processes. We just apply it for the treatment of Djordjević's experiment.

It should be emphasized that the subordination approach answers an important question: what a stochastic mechanism stands behind the combined law of relaxation in the studied system. We believe that this is caused by a multi-time scale subordination of random physical processes in such systems. Moreover, this is not the only physical development of multi-time scale subordination. In the next we consider how the subordination can influence on a state equation of continuous media.

4 State equation for continuous media with memory

Any continuous medium at any point $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$ and any time t is determined by the velocity vector $\mathbf{v}(\mathbf{x}, t) = (v_1, v_2, v_3)$, the pressure $p(\mathbf{x}, t)$ and the density $\rho(\mathbf{x}, t)$. The characteristic quantities are connected with each others by means of the Navier-Stokes equation

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p = \mu \Delta \mathbf{v},$$

the continuum equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0$$

and the state relation

$$F(p, \rho, \mathbf{v}, \mathbf{x}, t) = 0,$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$ is the Hamiltonian operator, μ the viscosity, Δ the Laplace operator, and F is an operator (see [27]).

Following [28], for a start we assume that the relaxation process in a continuous medium obeys a simple exponential law

$$dh/dt = -(h - h_0)/\theta \quad \text{or} \quad h = h_0 + (h(0) - h_0)e^{-t/\theta}, \quad (17)$$

where we do not define concretely the nature of relaxation, and θ is the relaxation time of the parameter h characterizing this medium. Next, a wave (sound or something like) is propagated in the medium. If the wave period is $T \gg \theta$, the media has time to accommodate oneself to wave changes, and the wave will advance with the velocity $c_0^2 = (\partial p / \partial \rho)_{h_0}$ (here h_0 corresponds to a value of h in this wave). If on the contrary $T \ll \theta$, then the parameter h will be “frozen” to its value h_{00} without any wave, and the wave velocity is $c_\infty^2 = (\partial p / \partial \rho)_{h_{00}}$. Suppose that the wave results in small perturbations of the density $\rho = \rho_0 + \rho'$ and the pressure $p = p_0 + p'$ for the given medium.

Let the state relation take the form $p = p(\rho, h)$. Expand it in a series about small perturbations:

$$p = p(\rho, h) \approx p_0 + \left(\frac{\partial p}{\partial \rho} \right)_{h_0} (\rho - \rho_0) + \left(\frac{\partial p}{\partial h} \right)_{\rho_0} (h - h_0). \quad (18)$$

As the equilibrium state has $h_0 = g_0(\rho)$, therefore

$$p(\rho, h_0) \approx p_0 + \left(\frac{\partial p}{\partial \rho} \right)_{h_{00}} (\rho - \rho_0) + \left(\frac{\partial p}{\partial h} \right)_{\rho_0} (h_0 - h_{00}). \quad (19)$$

It follows from this that

$$\left(\frac{\partial p}{\partial \rho} \right)_{h_0} = \left(\frac{\partial p}{\partial \rho} \right)_{h_{00}} + \left(\frac{\partial p}{\partial h} \right)_{\rho_0} \frac{\partial h_0}{\partial \rho}. \quad (20)$$

Next, we substitute the expression (20) to the expansion (18) and differentiate the obtained result with respect to t . Using the equation (17) one can find

$$\frac{dp'}{dt} = c_\infty^2 \frac{d\rho'}{dt} - \left(\frac{\partial p}{\partial g} \right)_{\rho_0} \frac{h - h_0}{\theta}. \quad (21)$$

The expression (21) shows a connection between increments of p' and ρ' in the wave and the deviation of the parameter $h - h_0$ from its equilibrium value. In order to get an equation depending only on p' and ρ' , we divide Eq.(18) on τ and sum the result with Eq.(21). Thus we arrive at the sought-for equation:

$$\frac{dp'}{dt} + \frac{p'}{\theta} = c_\infty^2 \frac{d\rho'}{dt} + c_0^2 \frac{\rho'}{\theta}. \quad (22)$$

It is not difficult to establish that this equation is equivalent to the integral relation

$$p' = c_0^2 \rho' + (c_\infty^2 - c_0^2) \int_{-\infty}^t \exp\left(-\frac{t-t_1}{\theta}\right) \frac{d\rho'}{dt_1} dt_1. \quad (23)$$

From Eq.(23) it is seen that the medium has memory effects.

For the derivation of Eq.(23) we accepted the relaxation equation of type (17). As a result, the kernel of Eq.(23) has the exponential form. Generally, the relaxation equation can be more complex (for example, for polymers it is non-exponential), and therefore the corresponding kernel may be other. In particular, for fractional systems it is written as a power function, and the corresponding state equation takes the fractional form of differential equations. Consider the feature in more details below.

A broad class of continuous media with a long-term memory (for example, polymer fluids, viscoelastic materials, etc. [29,30]) is described more adequately by the state equation in the fractional form of type

$$\frac{\partial^\alpha p_\alpha}{\partial t^\alpha} + \frac{p_\alpha}{\theta_1} = c_\infty^2 \frac{\partial^\beta \rho_\beta}{\partial t^\beta} + c_0^2 \frac{\rho_\beta}{\theta_1}, \quad (24)$$

where $\partial^\alpha/\partial t^\alpha$ is the fractional derivative in the sense of Caputo [12]. Here c_0 is the wave velocity, when the wave period is more than relaxation time θ_1 , and c_∞ is the wave velocity in the opposite case. To obtain Eq.(24) from Eq.(22), we present the pressure and the density as a result of subordination, namely

$$p_\alpha(\mathbf{x}, t) = \int_0^\infty f_\alpha(\tau_1, t) p(\mathbf{x}, \tau_1) d\tau_1, \quad (25)$$

$$\rho_\beta(\mathbf{x}, t) = \int_0^\infty f_\beta(\tau_2, t) \rho(\mathbf{x}, \tau_2) d\tau_2, \quad (26)$$

where $f_\alpha(\tau_1, t)$ (or $f_\beta(\tau_2, t)$) is the probability density of the inverse-time α (or β)-stable Lévy subordinator with $0 < \alpha, \beta < 1$. In other words, both the pressure and the density of such a medium are subordinated in different ways. From the physical point of view the subordination accounts that there are temporal (random) intervals, when the pressure and the density does not change. In any other case the random jumps in density and pressure occur. In particular, if $\alpha, \beta = 1$, then the values change continuously.

According to [19], the Laplace transform in time, for the pressure and the density in the medium with the long-term memory, gives

$$\bar{p}_\alpha(\mathbf{x}, s) = s^{\alpha-1} \bar{p}(\mathbf{x}, s^\alpha), \quad \bar{\rho}_\beta(\mathbf{x}, s) = s^{\beta-1} \bar{\rho}(\mathbf{x}, s^\beta). \quad (27)$$

Hence, Eq.(22) in the Laplace space reads

$$\begin{aligned} s\bar{p}(\mathbf{x}, s) - p(\mathbf{x}, 0) + \frac{\bar{p}(\mathbf{x}, s)}{\theta_1} = \\ = c_\infty^2 s\bar{\rho}(\mathbf{x}, s) - c_\infty^2 \rho(\mathbf{x}, 0) + c_0^2 \frac{\bar{\rho}(\mathbf{x}, s)}{\theta_1}. \end{aligned} \quad (28)$$

The subordination in different ways means that we should accept

$$\begin{aligned} \int_0^\infty \left(\frac{\partial p(\mathbf{x}, \tau_1)}{\partial \tau_1} + \frac{p(\mathbf{x}, \tau_1)}{\theta_1} \right) f_\alpha(\tau_1, t) d\tau_1 = \\ = \int_0^\infty \left(c_\infty^2 \frac{\partial \rho(\mathbf{x}, \tau_2)}{\partial \tau_2} + c_0^2 \frac{\rho(\mathbf{x}, \tau_2)}{\theta_1} \right) f_\beta(\tau_2, t) d\tau_2. \end{aligned} \quad (29)$$

Then the Laplace transform of the latter expression takes the form

$$\begin{aligned} s^{\alpha-1} (s^\alpha \bar{p}(\mathbf{x}, s^\alpha) - p(\mathbf{x}, 0)) + \frac{\bar{p}_\alpha(\mathbf{x}, s)}{\theta_1} = \\ c_\infty^2 s^{\beta-1} (s^\beta \bar{\rho}(\mathbf{x}, s^\beta) - \rho(\mathbf{x}, 0)) + c_0^2 \frac{\bar{\rho}_\beta(\mathbf{x}, s)}{\theta_1}. \end{aligned} \quad (30)$$

After simple algebraic transformations we have

$$s^\alpha \bar{p}_\alpha(\mathbf{x}, s) - s^{\alpha-1} p(\mathbf{x}, 0) + \frac{\bar{p}_\alpha(\mathbf{x}, s)}{\theta_1} = c_\infty^2 s^\beta \bar{\rho}_\beta(\mathbf{x}, s) - c_\infty^2 s^{\beta-1} \rho(\mathbf{x}, 0) + c_0^2 \frac{\bar{\rho}_\beta(\mathbf{x}, s)}{\theta_1}. \quad (31)$$

and the Laplace inverse of this expression leads directly to Eq.(24). The subordination of type (25) and (26) does not change the form of the Navier-Stokes equation, but in the continuum equation the time derivative for the density becomes fractional.

Finally, we consider the situation where both the Navier-Stokes equation and the continuum equation are one-dimensional, namely

$$\rho_0 \frac{\partial v}{\partial t} + \frac{\partial p_\alpha}{\partial x} = \mu \frac{\partial^2 v}{\partial x^2}, \quad v \equiv v_1, \quad x \equiv x_1, \quad (32)$$

$$\frac{\partial^\beta \rho_\beta}{\partial t^\beta} + \rho_0 \frac{\partial v}{\partial x} = 0, \quad \rho \equiv \text{const} > 0. \quad (33)$$

For $\alpha = \beta$ and $c_0 \ll 1$ Eq.(24) tends to the following expression

$$\theta_1 \frac{\partial^\alpha p_\alpha}{\partial t^\alpha} + p_\alpha = \theta_1 c_\infty^2 \frac{\partial^\alpha \rho_\alpha}{\partial t^\alpha}. \quad (34)$$

Let the viscosity μ be equal to zero. From Eqs.(32) and (33) we have

$$\frac{\partial}{\partial t} \frac{\partial^\alpha \rho_\alpha}{\partial t^\alpha} = \frac{\partial^2 p_\alpha}{\partial x^2}.$$

Using the state equation (34), the equation for the pressure takes the form

$$\frac{\partial p_\alpha}{\partial t} + \theta_1 \frac{\partial}{\partial t} \frac{\partial^\alpha p_\alpha}{\partial t^\alpha} = \theta_1 c_\infty^2 \frac{\partial^2 p_\alpha}{\partial x^2}. \quad (35)$$

The presented case is interesting because it leads to the fractional wave-diffusion equation (35), although the Navier-Stokes equation is not fractional. The fractional derivative appears in Eq.(35), due to the fractional state equation and the fractional continuum equation.

5 Conclusions

We have shown how the combined law of relaxation can be derived from the diffusion model based on the CTRW consideration and illustrated the role of

the two-time scale subordination. Our analysis demonstrates that the type of relaxation function depends entirely on the subordinator. The subordinators may be different and enough complicated. If the operational time is an inverse-time Lévy α -stable subordinator, the relaxation function becomes Mittag-Leffler's. The operational time of type $\bar{V}_t^{(\alpha)}$ “transforms” the relaxation function into the stretched exponential form. The multi-scale subordinator allows one to combine some laws of relaxation as their product. The result supports the conclusion that the subordination as a transformation from the physical time t to the operational time τ is responsible for the anomalous properties of complex systems. Using this approach to a similar problem in continuous mechanics, we notice a special role of the state relation. Due to the multi-scale subordination the relation takes the fractional form. Consequently, the continuum equation becomes mathematically more complicated than one in the ordinary mechanics.

Acknowledgments

AS is grateful to the Institute of Physics and the Hugo Steinhaus Center for Stochastic Methods for pleasant hospitality during his visit in Wrocław University of Technology.

References

- [1] D. Toussaint, F. Wilczek, J. Chem. Phys. 78 (1983) 2642-2647.
- [2] M. Bramson, J.L. Lebowitz, Phys. Rev. Lett. 61 (1988) 2397-2400.
- [3] D. ben Avraham, S. Havlin, Diffusion and Reactions in Fractals and Disordered Systems, Cambridge University Press, Cambridge, 2000.
- [4] J. Sung, E. Barkai, R.J. Silbey, S. Lee, J. Chem. Phys. 116(6) (2002) 2338-2341.
- [5] M. Moreau, G. Oshanin, O. Bénichou, M. Coppey, Phys. Rev. E 67 (2003) 045104R.
- [6] A.J. Bray, S.N. Majumdar, R. A. Blythe, Phys. Rev. E 67 (2003) 060102.
- [7] S.B. Yuste, K. Lindenberg, Phys. Rev. E 72 (2005) 061103.
- [8] S.B. Yuste, K. Lindenberg, Proceedings of SPIE 5845 (2005) 27-39.
- [9] J. Klafter, A. Blumen, G. Zumofen, J. Stat. Phys. 36 (1984) 561-577.
- [10] R. Metzler, J. Klafter, Phys. Rep. 339 (2000) 1-77.

- [11] W.R. Schneider, W. Wyss, J. Math. Phys. 30 (1989) 134-144.
- [12] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
- [13] M.M. Meerschaert, H.-P. Scheffler, J. Appl. Probab. 41 (2004) 623-638.
- [14] A.A. Stanislavsky, Theor. Math. Phys. 138(3) (2004) 418-431.
- [15] I.M. Sokolov, Phys. Rev. E 63 (2000) 011104.
- [16] A.A. Stanislavsky, Acta Phys. Polon. B 34(7) (2003) 3649-3660.
- [17] M. Magdziarz, K. Weron, Physica A 367 (2006) 1-6.
- [18] M. Magdziarz, K. Weron, Acta Phys. Polon. B 37(5) (2006) 1617-1624.
- [19] A.A. Stanislavsky, Phys. Rev E 67 (2003) 021111.
- [20] Z.B. Djordjević, Scaling observation in the trapping reaction, in: Fractals in Physics, Proceeding of Sixth Trieste International Symposium, Eds. L. Pietronero, E. Tossati, North-Holland, Amsterdam, 1986, pp. 581-585.
- [21] N.G. van Kampen, Stochastic processes in physics and chemistry, North-Holland, Amsterdam, 1987, Ch.V.
- [22] A.K. Jonscher, Dielectric Relaxation in Solids, Chelsea Dielectrics Press, London, 1983.
- [23] A.K. Jonscher, Universal Relaxation Law, Chelsea Dielectrics Press, London, 1996.
- [24] A. Jurlewicz, K. Weron, Cell. & Molec. Biol. Lett. 4 (1999) 55-86.
- [25] A. Piryatinska, A.I. Saichev, W.A. Woyczynski, Physica A 349 (2005) 375-420.
- [26] S.N. Majumdar, S.J. Cornel, Phys. Rev. E 57 (1998) 3757-37xx.
- [27] L.D. Landay, E.M. Lifshitz, Fluid Mechanics, Butterworth-Heinemann, Oxford, 1997.
- [28] M.B. Vinogradova, O.V. Rudenko, A.P. Sukhorukov, Wave Theory, Fiziko-matematicheskaya literatura, Moscow, 1990, pp. 87-90.
- [29] R.L. Bagley, P.J. Torvik, J. of Rheology 27(3) (1983) 201-210; P.J. Torvik, R.L. Bagley, J. Appl. Mech. 51 (1984) 294-298.
- [30] A.M. Nakhushiev, Fractional calculus and its application, Fiziko-matematicheskaya literatura, Moscow, 2005, pp. 189-193.